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THE EFFECT OF MERIDIONAL ELECTRIC VORTEX FLOW ON THE AZIMUTHAL ROTATION OF A FLUID†

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New exact solutions of the Navier–Stokes equations are obtained for spiral axisymmetric flow of a conducting fluid in bounded and unbounded regions. Attention is devoted to the influence of the poloidal component of the velocity field, generated by the meridional electric vortex flow, on the toroidal component due to the rotating boundaries of the region. A two-parameter family of self-similar solutions obtained by numerical integration of a system of non-linear ordinary differential equations is investigated. It is shown, considering twisted flow around a cylinder in an unbounded region and differential rotation between coaxial cylinders, that boundary layer regimes of meridional flow induce a boundary layer structure in the azimuthal rotation of the fluid.

SPIRAL vortex structures in fluids are of interest in connection with phenomena observed when magnetic fields are excited by moving conducting media (MHD-dynamos), in the formation of large-scale atmospheric eddies, the phenomenon of reverse energy cascade in turbulence, etc. In magnetohydrodynamics, three-dimensional vortex flows and magnetic fields are conveniently split (depending on the phenomenon under consideration) into mutually interacting toroidal and poloidal components [1]. Electric vortex (EV) flows, which are created by the interaction of a non-uniform electric current and an intrinsic magnetic field, are of particular interest in MHD. When EV flows are investigated in axially symmetric situations, one can find self-similar solutions of the MHD equations. In that case, however, only poloidal flows are possible. Toroidal flows, set up in the absence of external magnetic fields by azimuthal currents only, are not observed, since in an axially symmetric situation the φ-component of the electric field may arise neither from the action of external sources nor by induction from the motion of the fluid [2]. To organize a spiral structure, the

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azimuthal rotation of the fluid must have some other origin, such as rotating boundaries of the fluid region.

The aim of this paper is to investigate the effect of steady meridional EV flow on the twisted flow of an incompressible conducting fluid, over a wide range of Reynolds numbers.

1. STATEMENT OF THE PROBLEM

In an axisymmetric situation, the solenoidal property of the velocity field and magnetic field enables them to be described in cylindrical coordinates by three functions: $\psi_1 = \psi_1(r, z)$—the Stokes stream function, $\psi_2 = \psi_2(r, z)$—the current and $\psi_3 = \psi_3(r, z)$—the azimuthal velocity:

$$\mathbf{v} = (v_z, v_r, v_\varphi) = \left(\frac{1}{r}\frac{\partial\psi_1}{\partial r}, -\frac{1}{r}\frac{\partial\psi_1}{\partial z}, \frac{\psi_3}{r}\right), \quad \mathbf{B} = \left(0, 0, \frac{\mu_0\psi_2}{r}\right) \tag{1.1}$$

Using this representation, we can write the equations of magnetohydrodynamics in dimensionless form:

$$\begin{gathered}
\frac{\partial E^2\psi_1}{\partial t} + 2\eta\left\{\psi_1, \frac{E^2\psi_1}{\eta}\right\} + \frac{\alpha}{\eta}\frac{\partial\psi_3{}^2}{\partial\zeta} = E^4\psi_1 + \frac{S}{\eta}\frac{\partial\psi_2{}^2}{\partial\zeta} \\
\frac{\partial\psi_2}{\partial t} + 2\eta\left\{\psi_1, \frac{\psi_2}{\eta}\right\} = \frac{1}{\beta}E^2\psi_2, \quad \frac{\partial\psi_3}{\partial t} + 2\{\psi_1, \psi_3\} = E^2\psi_3 \\
\eta = \frac{r^2}{l^2}, \quad \zeta = \frac{z}{l}, \quad E^2 = 4\eta\frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial\zeta^2}, \quad \alpha = \frac{A_3{}^2 l}{\nu A_1} \\
\{F, G\} = \frac{\partial F}{\partial\eta}\frac{\partial G}{\partial\zeta} - \frac{\partial F}{\partial\zeta}\frac{\partial G}{\partial\eta}, \quad \beta = \mu_0\sigma\nu, \quad S = \frac{\mu_0 A_2{}^2 l}{\rho\nu A_1}
\end{gathered} \tag{1.2}$$

The scales used here are as follows: length l, time l^2/ν, functions ψ_i–A_i ($A_1 = \nu l$, $A_2 = I$, $A_3 = \nu$); ρ denotes the density, ν the kinematic viscosity, σ the conductivity of the fluid and I some characteristic electric current in the fluid.

Consider the steady axisymmetric flow of an electric current to an infinite dielectric cylinder of radius r_0 aligned along the z axis. We will examine two different geometries of the region occupied by the fluid: (1) the electric current is flowing in an unbounded conducting fluid and (2) the region of flow is bounded by a coaxial solid mass of the same conductivity as the fluid.

In cylindrical coordinates, there are several ways to separate the variables in system (1.2). For the situations considered here we have the representation

$$\psi_1 = \zeta f_1(\eta), \quad \psi_2 = \zeta f_2(\eta), \quad \psi_3 = f_3(\eta) \tag{1.3}$$

that is, the azimuthal rotation of the fluid is independent of ζ. Substituting (1.3) into (1.2) we obtain a coupled system of non-linear ordinary differential equations for $f_i(\eta)$ (the prime denotes differentiation with respect to η):

$$\frac{\partial f_1''}{\partial t} + 2(f_1' f_1'' - f_1 f_1''') = 4\eta f_1^{\mathrm{IV}} + 8f_1''' + \frac{S}{2}\frac{f_2{}^2}{\eta^2} \tag{1.4}$$

$$\frac{\partial f_2}{\partial t} + 2\left(f_2 f_1' - f_2' f_1 + \frac{f_1 f_2}{\eta}\right) = \frac{4}{\beta}\eta f_2'', \quad \frac{\partial f_3}{\partial t} + 2f_1 f_3' = 4\eta f_3'' \tag{1.5}$$

In the context of electric vortex flows, allowance should be made for the fact that in real conducting fluids the Batchelor number is negligibly small: $10^{-7} < \beta < 10^{-6}$. Therefore, expanding the azimuthal magnetic field [in our notation—the function $f_2(\eta)$] in powers of β, one usually confines ones attention to terms of the zeroth or first order (the electrodynamic or non-inductive approximation) [2]. However, when investigating EV flows in an unbounded volume of conducting fluid, one must carefully stipulate the limits of applicability of the low magnetic Reynolds number (Re_m) condition if the components of the velocity field have a positive power dependence on the coordinates.

In the scheme adopted here for the flow-sheet of electric current, the current density ($j_r = -f_2/\sqrt{\eta}$, $j_z = 2\zeta f_2'$) satisfies the following boundary conditions: $f_2(\eta_0) = 0, f_2'(\infty) = 1$ ($\eta_0 = r_0^2/l^2$) [3]. The solution of the magnetic field induction equation (1.5) in the electrodynamic approximation is trivial: $f_2 = \eta - \eta_0$.

As a quantitative characteristic of the electric current, we take the total current I_0 flowing through the section of the cylindrical surface $\eta = 2\eta_0$ bounded by the planes $\zeta = \pm 1/\eta_0$. Then $I = I_0/(4\pi)$ and $S = \mu_0 I_0^2/[(4\pi\nu)^2\rho]$. The streamlines of the electric current are hyperbolic and symmetrical about the plane $\zeta = 0$.

The three-dimensional non-uniformity of the current leads to the formation of a vortical component of the Lorentz force [the last term in Eq. (1.4)], thus creating a flow in the fluid. The parameter S characterizes the rate of the induced EV flow and is an analogue of the Reynolds number. The meridional motion of the fluid is independent of the direction of the electric field, since $S \sim I_0^2$. As we shall show below, although the vorticity distribution of the electromagnetic force remains fixed in both cases (both for flow outside the cylinder in an unbounded fluid and for flow between coaxial cylinders), the poloidal circulation of the fluid and its interaction with the azimuthal rotation are in opposite directions.

2. TWISTED FLOW NEAR THE CRITICAL CIRCLE

Suppose that a dielectric cylinder of radius $r_0 = 2l$ is rotating at angular velocity $\Omega = \nu/(4l^2)$ in an unbounded conducting fluid. The dimensionless azimuthal velocity $v_\varphi = f_3/\sqrt{\eta}$ must satisfy the boundary condition

$$f_3 = 1 \text{ at } \eta = \eta_0 = 4 \tag{2.1}$$

Physical considerations imply that the following boundary conditions are possible at infinity:

$$\text{(a) } f_3 = 0, \text{ (b) } f_3 = 1, \text{ (c) } f_3' = \text{const} \tag{2.2}$$

The first condition means that there is no rotational motion of the fluid at an infinite distance from the cylinder, the second corresponds to differential rotation of the fluid, and the third to rigid rotation at $\Omega = \text{const}$.

If the cylinder surface is solid and impermeable, the no-slip condition must hold there:

$$f_1 = f_1' = 0 \text{ at } \eta = \eta_0 \tag{2.3}$$

If $S = 0$ the solution of the steady-state equation (1.4) will describe irrotational flow near the critical circle $\zeta = 0$, $\eta_0 = 4$ (the circle T at which the pressure of the fluid is a maximum) [4]:

$$f_1 = -6 + \eta + 2\exp(2 - \eta/2) \tag{2.4}$$

The fluid approaches the cylinder radially from infinity, flowing up and down its surface. An analytical solution of Eq. (1.4) exists only at $\eta_0 = 4$, owing to the choice of the cylinder radius as $r_0 = 2l$.

If $S \neq 0$, analysis of the electromagnetic forces shows that, beginning from some $\eta_* \gg \eta_0$, the curl of the driving force takes a constant value $S/4$, so that the problem has an exterior solution [3]:

$$\Phi = \left(\frac{S}{6}\right)^{1/2} \eta^{3/2} - 1 \tag{2.5}$$

which is the asymptotic limit of the solution of problem (1.4), (2.3):

$$f_1 \to \Phi, \ f_1' \to \Phi' \text{ as } \eta \to \infty \tag{2.6}$$

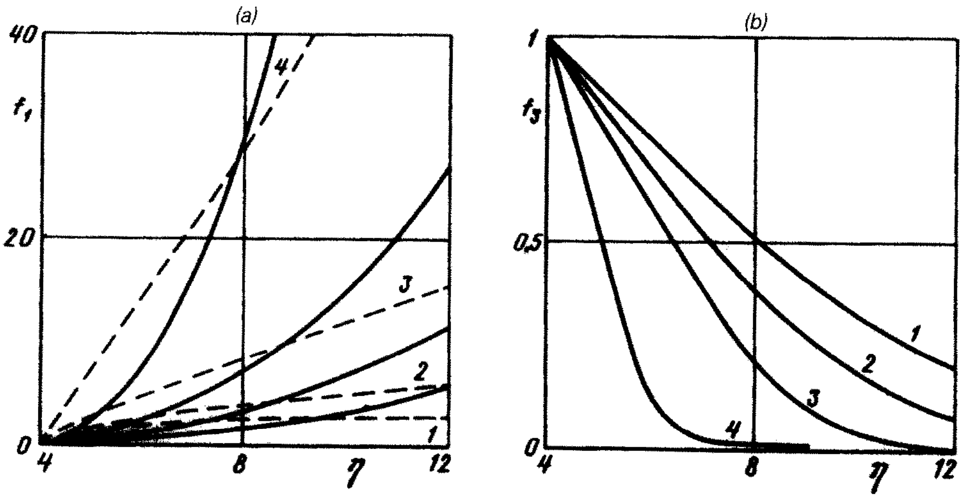

FIG. 1.

The one-parameter family of solutions of systems (1.4), (1.5), (2.1)–(2.3) and (2.5) has been investigated by the fifth-order Kutta–Merson method. The missing boundary conditions at $\eta = \eta_0$, which appear when the boundary-value problem is reduced to a Cauchy problem, are determined by Hooke–Jeeves optimization. The solution was considered to be adequate if it agreed with the asymptotic condition (2.6) at $\eta \gg \eta_0$ to within 0.01%. The numerical procedure was tested against the analytical solution (2.4).

Figure 1 shows the results of numerical integration at $S = 0, 1, 10, 100$ (curves 1–4, respectively). The radial profiles of the axial component $f_1' = v_z/(2\zeta)$ (the dashed curves) and radial component $f_1 = v_r\sqrt{\eta}$ (the solid curves) of the poloidal flow are shown in Fig. 1(a). Unlike the irrotational flow to the cylinder at $S = 0$, the velocity in EV flow has non-zero vorticity not only near the critical circle but even at infinity. Indeed, the viscous stress tensor component $\sigma_{\eta\zeta}' = 4\zeta\sqrt{\eta}\Phi''(\eta) = 3\zeta\sqrt{S/6}$ is constant at infinity for fixed ζ. Thus, self-similar solutions of the non-linear equations are characterized by the presence of a viscous core of the flow. In that case the region of flow near the critical circle in which the vorticity of the velocity, due to both viscous and electromagnetic forces, is appreciably different from its asymptotic value may be treated as a "boundary layer".

In steady flow, the equation for f_3 [see (1.5)] may be integrated twice taking into account condition (2.1):

$$f_3(\eta) = 1 + f_3'(\eta_0) \int_{\eta_0}^{\eta} \exp\left(-\int_{\eta_0}^{\eta} \frac{f_1}{2\eta}\, d\eta\right) d\eta \tag{2.7}$$

It can be shown that this solution cannot meet the third condition of (2.2) (rigid rotation of the bulk of the fluid outside the cylinder). When meridional EV flow is present, only differential rotation of the fluid is possible, $\Omega = \Omega(\eta)$.

The solution of Eq. (1.5) for f_3 satisfying the second condition of (2.2) is trivial: $f_3 = 1$, which is independent of the meridional circulation: the fluid rotates at constant angular velocity $\Omega = 1/\eta$. As for the first boundary condition in (2.2), the function $f_3(\eta)$ for various values of S is shown in Fig. 1(b). It is obvious that as the rate of EV flow increases the azimuthal rotation is localized near the surface of the cylinder and a boundary-layer structure is formed.

3. DIFFERENTIAL ROTATION BETWEEN COAXIAL CYLINDERS

We will now consider the effect of meridional EV flow on the azimuthal motion of a fluid between two infinite coaxial cylinders rotating about their axes at angular velocities Ω_1 and Ω_2. Let $R_1 = l$ be the radius of the inner dielectric cylinder, R_2 that of the outer, conducting cylinder, or, in new variables, $\eta_1 = 1$ and $\eta_2 = 1 + \delta$, where $e\delta$ is the distance between the cylinders. We introduce dimensionless angular velocities $\omega_i = \Omega_i l^2/\nu$, $i = 1, 2$. At the cylinder surfaces we require the no-slip conditions to hold:

$$f_1=0,\ f_1'=0,\ f_3=\omega_1\eta_1 \quad \text{at} \quad \eta=\eta_1$$
$$f_1=0,\ f_1'=0,\ f_3=\omega_2\eta_2 \quad \text{at} \quad \eta=\eta_2 \tag{3.1}$$

When there is no electric vortex flow ($S = 0$) the last equation of (1.5) describes classical Couette flow between rotating cylinders [5]. Note that if the angular velocity ratio is $\omega_2/\omega_1 = \eta_1/\eta_2$, then f_3 = const and the azimuthal rotation is independent of EV flow.

The magnitude of the EV flow parameter S for flow in an unbounded region was not of decisive value, since for any S, however small, the Reynolds number $\mathrm{Re} = v_{\max}(\eta)l/\nu = \eta\sqrt{S/6}$ [see (2.5)] may become as large as desired with increasing distance from the cylinder; in the boundary-value problem (3.1), however, the magnitude of the eigenvalue S determines the regime of EV flow.

At small S (creeping flow), the linear solution of problem (1.4), (3.1) is proportional to the EV flow parameter:

$$f_1(\eta)=S(\alpha_1\eta^3+\alpha_2\eta^2+\alpha_3\eta+\alpha_4\eta\ln\eta-\eta^2\ln\eta-\eta\ln^2\eta/2+\alpha_5) \tag{3.2}$$

$$\begin{gathered}
\alpha_0=12[2(1-\eta_1)^2+(1-\eta_1^2)\ln\eta_1],\quad \alpha_1=1/12\\
\alpha_2=-[3(1-\eta_1)^2(\eta_1-3)+2(1-\eta_1)(\eta_1^2+7\eta_1+1)\ln\eta_1+\\
+6(2\eta_1^2+\eta_1+1)\ln^2\eta_1]/\alpha_0\\
\alpha_3=[(1-\eta_1)^2(\eta_1^2-8\eta_1-11)+(1-\eta_1)(4\eta_1^2+25\eta_1+1)\ln\eta_1+\\
+6(5\eta_1^2+2\eta_1+1)\ln^2\eta_1]/\alpha_0\\
\alpha_4=[(1-\eta_1)^2(11+14\eta_1-\eta_1^2)+12(1-\eta_1^2)\ln\eta_1+\\
+6(1-\eta_1^2)\ln^2\eta_1]/\alpha_0\\
\alpha_5=-[(1-\eta_1)^2(\eta_1^2-11\eta_1)+2\eta_1(1-\eta_1)(\eta_1+6)\ln\eta_1+\\
+6\eta_1(3\eta_1+1)\ln^2\eta_1]/\alpha_0
\end{gathered}$$

According to (3.2), (1.1), EV flow between cylinders takes the form of bitoroidal flow which is symmetrical with respect to the plane $\zeta = 0$; in the upper vortex the fluid circulates in the counterclockwise sense.

As S increases, the non-linearity of Eq. (1.4) comes to the fore. Electromagnetic forces are balanced by viscous forces owing to the large gradients near the solid boundaries; in the inviscid core of the flow they are balanced by the inertial terms. The essentially non-linear nature of EV flow is indicated by the proportionality to $\sqrt{S}$.

Numerical analysis of the problem described by Eq. (1.4), the second equation of (1.5) and conditions (3.1) was carried out, with the geometrical parameter fixed at $\delta = 1$, giving S the successive values 2×10^3, 10^5, 10^6, 5×10^8 (curves 1–4 in Fig. 2). The solid curves in Fig. 2(a) are profiles of the radial velocity component ($F = v_r/\sqrt{S}$) while the dashed curves are those of the axial component ($F' = v_z/(2\zeta\sqrt{S})$. We see that as S increases the vortex centres are displaced toward the surface of the outer cylinder. The φ-component of the velocity vorticity ($\Gamma = -4\zeta f_1''\sqrt{\eta/S}$) is shown as a function of η in Fig. 2(b). In the Stokes regime EV flow takes place symmetrically between the cylinders (curve 1). As S increases the friction increases more slowly at the surface of the dielectric cylinder than at that of the conducting one. When $S>10^6$ (curve 4) the EV flow is deformed in the direction of the outer cylinder, with the formation of a pronounced boundary layer.

At $\omega_1 = 0$, $\omega_2 = 1$ (only the outer cylinder is rotating), the classical formula of Couette flow [5] gives $f_3 = (1+\delta)(\eta-1)/\delta$ (Fig. 3a, curve 1, corresponding to $S = 0$). At other S values: 10^5, 6×10^6 and 5×10^8 (curves 2–4, respectively), the radial function of the azimuthal velocity $f_3(\eta)$ changes appreciably, indicating that the rotation of the fluid is localized near the surface of the rotating cylinder.

At $\omega_1 = 0.2$, $\omega_2 = -0.8$, the rotation inversion point ($v_\varphi\equiv0$) is displaced as S increases to the outer boundary (Fig. 3b).

Let us investigate the effect of the essentially non-linear EV flow regime on rotating motion in a cylinder layer in the general case $\omega_2 = \alpha\omega_1$. Figure 4 shows the radial angular velocity function of the fluid, $\omega = \omega(\eta)$, at $S = 6\times10^6$, $\omega_1 = 1$ and α is equal respectively to -10, -1.5, 0, 0.5, 1, 5, 50 (curves 1–7). The region of twisted

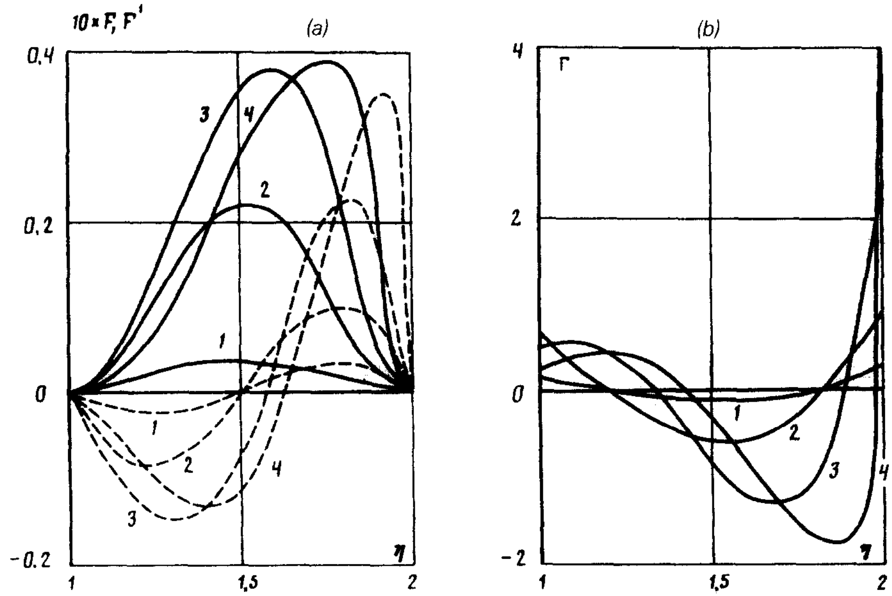

FIG. 2.

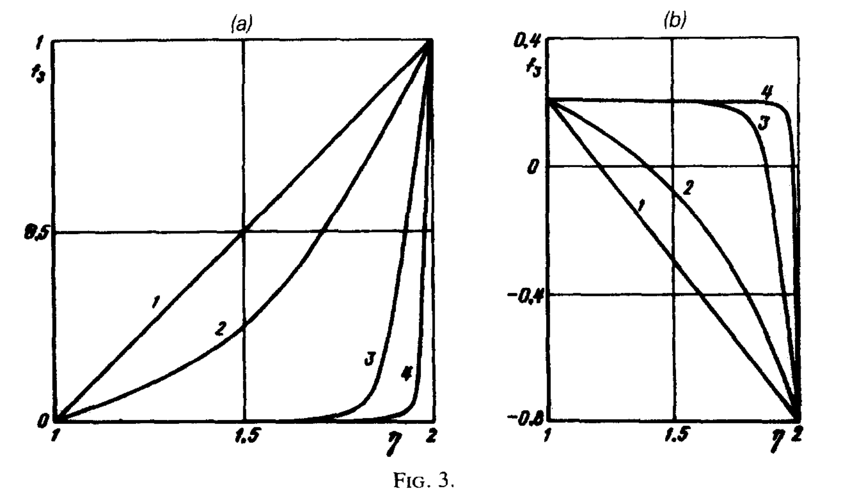

FIG. 3.

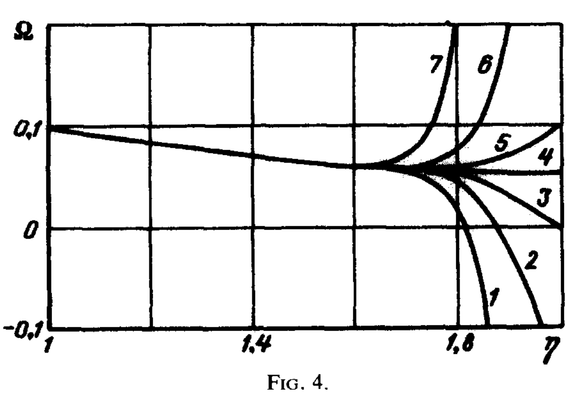

FIG. 4.

flow is divided into a region $\eta_1 \leqslant \eta \leqslant \eta_*(S)$, in which the fluid rotates at an angular velocity ω_1/η ("rotation core") and a region $\eta_*(S) \leqslant \eta \leqslant \eta_2$ in which $\omega(\eta)$ depends on α.

Thus, in both cases considered (flow around a cylinder in an unbounded region and flow between coaxial cylinders), the well-developed meridional boundary-layer flow induces a boundary layer structure in the azimuthal rotation of the fluid.

Another characteristic feature of axisymmetric EV flows is that, in a given field of body forces, one can alter the flow structure—the direction of meridional circulation and the position of the boundary layers. This may be used to intensify the differential rotation of a conducting fluid. Regulation of the flow of liquefied metals in an annular gap is used in lubrication, centrifugal founding, etc. The boundary layer structure of rotational motion has found application in MHD separation for ore dressing, the purification of liquid metals, and the extraction of metals and oxides from slags [6].

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